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Discrete Gradients in Discrete Classical Mechanics

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A simple model of discrete classical mechanics is given where, starting from the continuous Hamilton equations, discrete equations of motion are established together with a proper discrete gradient definition. The conservation laws of the total discrete momentum, angular momentum, and energy are demonstrated.

1. INTRODUCTION

The problem of the motion of complex physical material systems (formed by more than two particles) is usually solved by means of numerical calculations. In continuous classical mechanics the dynamical behavior of the material systems is determined by continuous dynamical variables which are solutions of the differential equations of the motion. The equations of motion are approximated with difference equations and the dynamical variables with discrete functions of the time. In the numerical representation of the evolution of the physical systems the fundamental conservation laws are satisfied only approximately.

In *discrete* mechanics, instead, the discrete equations of motion retain, in general, the various symmetries and conservation laws. Therefore, discrete mechanics can ensure a significant improvement in numerical efficiency and satisfactory results can be more easily achieved.

Discrete classical mechanics differs from the continuous one in that the physical quantities and the dynamical variables are *defined* as discrete functions of the discrete time t_n, and one supposes that the equations of motion are convenient difference equations, which, in the limit $t_{n+1} - t_n = \Delta t_n \to 0$, reduce to the corresponding continuous differential equations. At each discrete time instant t_n, the fundamental conservation laws must be automatically satisfied. Hence, discrete equations must lead to the same

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laws of continuous mechanics. This suggests that the discrete equations of motion have a formal analogy with the known equations of classical mechanics.

These considerations pose a limit on the possibility of building discrete mechanics models.

Discrete classical mechanics finds its motivations in the simplicity of the algebraic structure of the equations of motion, whose solution sometimes demands only simple operations of an arithmetical nature. The difficulty of obtaining analytical solutions in some cases is largely compensated by the high computability. As pointed out above, complicated continuous dynamical systems are frequently solved numerically by discretizing them; in discrete mechanics the numerical solutions are strictly related to the continuous solutions and the validity of the conservation laws can avoid numerical instabilities. In additions, discrete mechanics allows one to gain insight into the understanding of the physical problem considered, and finds applications in many fields of physics.

In this paper we limit ourselves to a simple model of discrete classical mechanics. Our approach is based on a correspondence between *analytical* and *discrete* Hamilton equations. The equations of motion are hence obtained by defining a convenient discrete gradient in analogy with the continuous one. Our results are general enough and impose no strict limitations on the expression of the force between the interacting particles.

Recently, Lee (1983) has proposed a very interesting new formulation of mechanics where time is considered as a discrete dynamical variable and the usual continuous mechanics appears as an approximation. His discrete mechanics finds its greatest applications in quantum and relativistic mechanics and may also be regarded as a possible way for the elimination of the divergence difficulties of quantum field theory (Friedberg and Lee, 1983). The consequences in quantum mechanics of a Lagrangian different from that used by Lee are discussed by D'Innocenzo *et al.* (1984). Lee's discrete mechanics is conceptually different from the model proposed in this work, where the time remains a parameter, although discrete.

Another model of discrete mechanics, based on the discretization of the forces between the interacting particles has been proposed (Greenspan, 1974). In this latter model, for each physical system discrete forces were chosen in a suitable way in order to achieve the validity of the conservation laws, but a general procedure for finding them was not given. On the contrary, in our model, discrete force expressions arise in a quite natural way from the discrete Hamilton equations, independently of the specific kind of physical system considered.

In Section 2 the discrete equations of motion are established by a formal analogy with the corresponding continuous equations. In Section 3 the generalized discrete gradients are defined in a convenient way and are used in Section 4 in order to demonstrate the validity of the conservation laws of the total momentum, energy, and angular momentum. Finally, in Section 5, the foregoing results are applicated to simple physical problems.

2. DISCRETE EQUATIONS OF MOTION

In continuous mechanics a dynamical system of k particles is described by a Hamiltonian H, and the position vector $\mathbf{r}^{(i)}$ of the ith particle and its respective momentum $\mathbf{p}^{(i)}$ obey the Hamilton equations. In the most common cases, the total energy $E = H$, and if t does not appear explicitly in H, the energy E is conserved. The conservation law of a dynamical variable that does not depend on the time explicitly can be formulated in terms of Poisson brackets. Thus, if s is a constant of the motion, it must have the property $[H, s] = 0$. Namely, the Poisson bracket of a motion constant with the Hamiltonian vanishes. This is the usual classical approach to continuous mechanics (Landau and Lifshitz, 1969).

In order to postulate the *discrete* equations of motion, we establish a correspondence between continuous and discrete variables and operators. The time is considered as a discrete parameter t_n, and the continuous functions $\mathbf{r}(t)$ and $\mathbf{p}(t)$ are thus replaced by the discrete quantities $\mathbf{r}_n = \mathbf{r}(t_n)$ and $\mathbf{p}_n = \mathbf{p}(t_n)$. Now, we can postulate the following correspondences:

$$\begin{aligned} &H \rightarrow H_n \equiv H(\mathbf{r}_n^{(1)}, \ldots, \mathbf{r}_n^{(k)}; \mathbf{p}_n^{(1)}, \ldots, \mathbf{p}_n^{(k)}, t_n) \\ &\partial/\partial \mathbf{v} \rightarrow \partial/\partial \mathbf{v}_n \end{aligned} \tag{1}$$

where H_n is the discrete Hamiltonian and $\partial/\partial \mathbf{v}_n$ is a discrete operator (it will be defined in the following) analogous to the continuous gradient operator.

We also postulate that the discrete equations of motion are

$$\dot{\mathbf{r}}_n^{(i)} = \partial H_n/\partial \mathbf{p}_n^{(i)}, \qquad i = 1, \ldots, k \tag{2}$$

and

$$\dot{\mathbf{p}}_n^{(i)} = -\partial H_n/\partial \mathbf{r}_n^{(i)}, \qquad i = 1, \ldots, k \tag{3}$$

where the derivative with respect to the time of the discrete vector $\mathbf{v}_n$ is defined as

$$\dot{\mathbf{v}}_n = \frac{\mathbf{v}_{n+1} - \mathbf{v}_n}{t_{n+1} - t_n} \equiv \frac{\Delta \mathbf{v}_n}{\Delta t_n} \tag{4}$$

We now require that the conservation laws remain valid also in discrete mechanics. The validity of the discrete conservation laws will depend on the choice of the discrete gradient. Thus, in analogy with the continuous case, we affirm that the energy of a physical system is a constant of the motion if the discrete Hamiltonian H_n does not depend on the time explicitly; in the same way the discrete variable s_n is a constant of the motion if the discrete Poisson bracket

$$[H_n, s_n] = \sum_{i=1}^{k} \left(\frac{\partial H_n}{\partial \mathbf{r}_n^{(i)}} \cdot \frac{\partial s_n}{\partial \mathbf{p}_n^{(i)}} - \frac{\partial H_n}{\partial \mathbf{P}_n^{(i)}} \cdot \frac{\partial H_n}{\partial \mathbf{r}_n^{(i)}} \right) \tag{5}$$

vanishes for every n.

3. GENERALIZED DISCRETE GRADIENTS

Let $f(\mathbf{r})$ be a scalar function of the vector $\mathbf{r}$. In the classical approach to differentiability the function $f(\mathbf{r})$ is approximated around $\mathbf{r}$ by a linear function $g(\mathbf{h})$ of the increment $\mathbf{h}$,

$$f(\mathbf{r}+\mathbf{h}) - f(\mathbf{r}) = g(\mathbf{h}) + o(|\mathbf{h}|) \tag{6}$$

where

$$g(\mathbf{h}) = h f'_{\mathbf{h}}(\mathbf{r}) \tag{7}$$

and $f'_{\mathbf{h}}(\mathbf{r})$ is the directional derivative of the function f at $\mathbf{r}$ with respect to the various vectors $\mathbf{h}$.

One defines the *gradient* of $\mathbf{r}$ as the unique $\mathbf{g}$ such that

$$g(\mathbf{h}) = \mathbf{g} \cdot \mathbf{h} \tag{8}$$

for all the vectors $\mathbf{h}$. From (8) one has

$$g(\mathbf{h}) = \mathbf{h} \cdot \nabla f(\mathbf{r})$$

and from (7)

$$f'_{\mathbf{h}}(\mathbf{r}) = g(\mathbf{h})/h = \hat{\mathbf{h}} \cdot \nabla f(\mathbf{r})$$

where $\hat{\mathbf{h}} = \mathbf{h}/h$ is the versor in the $\mathbf{h}$ direction.

As we have already said, we limit ourselves to scalar functions of $\mathbf{r}$, which depend only on $\mathbf{r}$, and we begin by observing that the gradient of such functions is

$$\nabla f(\mathbf{r}) = 2\mathbf{r}\, \partial f/\partial \mathbf{r}^2 \tag{9}$$

and therefore it has the direction of $\mathbf{r}$.

Now, in the attempt to define an analogue of the gradient for discrete functions, we require that it satisfies some requisites. In fact, the discrete gradient must have a formal analogy with the continuous gradient and must tend to the continuous one for $\Delta t_n \to 0$. Furthermore, it does lead to a discrete derivative in the one-dimensional case.

Based on these observations and looking to (9), we pose (this always can be done without any restrictions)

$$f(\mathbf{r}+\mathbf{h})-f(\mathbf{r})=G(\mathbf{r}+\mathbf{h},\mathbf{r})(2\mathbf{r}+\mathbf{h})\cdot\mathbf{h} \tag{10}$$

from which it follows that

$$G(\mathbf{r}+\mathbf{h},\mathbf{r})=\frac{f(\mathbf{r}+\mathbf{h})-f(\mathbf{r})}{(2\mathbf{r}+\mathbf{h})\cdot\mathbf{h}} \tag{11}$$

Taking a term from the language of mathematical programming (Clarke, 1975; Rockafellar, 1983), we define

$$\nabla_{\mathbf{h}}f=G(\mathbf{r}+\mathbf{h},\mathbf{r})(2\mathbf{r}+\mathbf{h}) \tag{12}$$

as the *generalized discrete gradient.*

The *discrete partial derivative* is given by

$$\partial_x f=\hat{\mathbf{x}}\cdot\nabla_{\mathbf{h}}f \tag{13}$$

and if f is a function of x only, it coincides with an incremental ratio, that is

$$\partial_x f(x)=\frac{\Delta f}{\Delta x}=\frac{f(x+\Delta x)-f(x)}{\Delta x} \tag{14}$$

For example, we have, from (14),

$$\partial_x x=1, \qquad \partial_x x^2=2x+h_x$$

where $h_x=\Delta x$.

Since $x=\mathbf{r}\cdot\hat{\mathbf{x}}$, we can pose

$$\nabla_{\mathbf{h}}x=\nabla_{\mathbf{h}}(\mathbf{r}\cdot\hat{\mathbf{x}})=\hat{\mathbf{x}} \tag{15}$$

and

$$\nabla_{\mathbf{h}}x^2=\nabla_{\mathbf{h}}(\mathbf{r}\cdot\hat{\mathbf{x}})^2=(2x+\mathbf{h}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}} \tag{16}$$

We notice that equation (15) can be written as

$$\nabla_{\mathbf{h}}(\mathbf{r}\cdot\hat{\mathbf{x}})=\hat{\mathbf{x}}\cdot\nabla_{\mathbf{h}}\mathbf{r} \tag{17}$$

Furthermore, we have in the continuum case

$$\nabla\mathbf{r}^2=\mathbf{r}\cdot\nabla\mathbf{r}+\mathbf{r}\cdot\nabla\mathbf{r} \tag{18}$$

and in the discrete case

$$\nabla_{\mathbf{h}}\mathbf{r}^2=2\mathbf{r}+\mathbf{h} \tag{19}$$

This equation can be written as

$$\nabla_{\mathbf{h}}\mathbf{r}^2 = \frac{2\mathbf{r}+\mathbf{h}}{2} \cdot \nabla_{\mathbf{h}}\mathbf{r} + \frac{2\mathbf{r}+\mathbf{h}}{2} \cdot \nabla_{\mathbf{h}}\mathbf{r} \tag{20}$$

which is the analogous discrete version of equation (18).

Generalizing these results, the discrete gradient of the component of $\mathbf{r}$ along the direction of a dynamical variable $\mathbf{p}$ is *assumed* to be

$$\nabla_{\mathbf{h}}(\mathbf{r} \cdot \mathbf{p}) = \tfrac{1}{2}(2\mathbf{p}+\mathbf{u}) \cdot \nabla_{\mathbf{h}}\mathbf{r} \tag{21}$$

that is,

$$\nabla_{\mathbf{h}}(\mathbf{r} \cdot \mathbf{p}) = (2\mathbf{p}+\mathbf{u})/2 \tag{22}$$

where $\mathbf{u}$ is the increment of the vector $\mathbf{p}$.

We observe that both $\mathbf{h}$ and $\mathbf{u}$ are determined by the discrete equations of motion (2) and (3).

From equation (16), which can be regarded as a discrete function $f(\mathbf{r} \cdot \hat{\mathbf{x}})$, we have the general rule

$$\nabla_{\mathbf{h}}f(\mathbf{r} \cdot \mathbf{p}) = \frac{\Delta f(\mathbf{r} \cdot \mathbf{p})}{\Delta(\mathbf{r} \cdot \mathbf{p})} \nabla_{\mathbf{h}}(\mathbf{r} \cdot \mathbf{p}) \tag{23}$$

Finally, we observe that equation (12) can also be written as

$$\nabla_{\mathbf{h}}f(\mathbf{r}) = 2\,\frac{2\mathbf{r}+\mathbf{h}}{2}\,\frac{\Delta f(\mathbf{r})}{\Delta \mathbf{r}^2} \tag{24}$$

Thus, many discrete operations preserve a formal analogy with the continuous ones.

4. CONSERVATION LAWS

We have already defined the discrete motion equations. In order to solve them for a particular problem, we must express the coordinates and momenta in terms of the intial conditions, i.e., the initial values $\mathbf{r}_0$ and $\mathbf{p}_0$ at the time $t = t_0$. In general, this will require the solution of $2N$ difference equations for a system with N degrees of freedom. Bypassing, for the moment, this problem, we shall show that, in our model, when the corresponding continuum physical conditions of validity are satisfied, the conservation laws of the total discrete energy, discrete momentum, and discrete angular momentum are automatically satisfied.

The energy of the system is conserved if the discrete Hamiltonian H_n does not explicitly depend on t_n. In fact, one has, utilizing equations (2) and (3),

$$\frac{\Delta H_n}{\Delta t_n} \equiv \frac{H_{n+1} - H_n}{t_{n+1} + t_n} = \partial_t H_n + \sum_i \left(\frac{\partial H_n}{\partial \mathbf{r}_n^{(i)}} \cdot \dot{\mathbf{r}}_n^{(i)} \right) = \partial_t H_n \tag{25}$$

Thus, if $\partial_t H_n = 0$, we obtain that the discrete energy is a motion constant

$$H_{n+1} = H_n \tag{26}$$

The total momentum of a system of k interacting particles

$$\mathbf{P}_n = \sum_{i=1}^{k} \mathbf{p}_n^{(i)} \tag{27}$$

is a motion constant if the Poisson bracket of the Hamiltonian and each of the components of P_n is equal to zero, namely

$$[H_n, P_{nx}] = 0 \tag{28}$$

By using equation (5), we have from this condition

$$\sum_{i=1}^{k} \frac{\partial H_n}{\partial \mathbf{r}_n^{(i)}} \cdot \hat{\mathbf{x}} = \sum_{i=1}^{k} F_{n_x}^{(i)} = 0 \tag{29}$$

which is satisfied if the internal discrete forces are Newtonian and the external resultant force is equal to zero.

Finally, let us define the discrete angular momentum of a particle at the discrete time t_n as

$$\mathbf{L}_n = \mathbf{r}_n \times \mathbf{p}_n \tag{30}$$

We demonstrate that

$$[\varphi, L_{n_z}] = 0 \tag{31}$$

where φ is any scalar function of the discrete coordinates and momentum of the particle and L_{nz} is the z component of the discrete angular momentum (30). Relation (31) is easily verified in continuous mechanics (Landau and Lifshitz, 1969).

Now, φ can only depend on the combinations $\mathbf{r}_n^2$, $\mathbf{p}_n^2$ and $\mathbf{r}_n \cdot \mathbf{p}_n$ of the vectors $\mathbf{r}_n$ and $\mathbf{p}_n$. Thus,

$$\frac{\partial \varphi}{\partial \mathbf{r}_n} = \frac{\Delta \varphi}{\Delta \mathbf{r}_n^2} (\mathbf{r}_{n+1} + \mathbf{r}_n) + \frac{\Delta \varphi}{\Delta(\mathbf{r}_n \cdot \mathbf{p}_n)} \frac{\mathbf{p}_{n+1} + \mathbf{p}_n}{2} \tag{32}$$

and

$$\frac{\partial \varphi}{\partial \mathbf{p}_n} = \frac{\Delta \varphi}{\Delta \mathbf{p}_n^2} (\mathbf{p}_{n+1} + \mathbf{p}_n) + \frac{\Delta \varphi}{\Delta(\mathbf{r}_n \cdot \mathbf{p}_n)} \frac{\mathbf{r}_{n+1} + \mathbf{r}_n}{2} \tag{33}$$

By using

$$L_{n_z} = \mathbf{r}_{nx} \mathbf{p}_n \cdot \hat{\mathbf{z}} = \mathbf{p}_n \times \hat{\mathbf{z}} \cdot \mathbf{r}_n = \hat{\mathbf{z}} \times \mathbf{r}_n \cdot \mathbf{p}_n \tag{34}$$

we have

$$\frac{\partial L_{n_z}}{\partial \mathbf{r}_n} = \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2} \cdot \frac{\partial}{\partial \mathbf{r}_n}(\hat{\mathbf{z}} \times \mathbf{r}_n) = \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2} \cdot \frac{\partial}{\partial \mathbf{r}_n}(x_n\hat{y} - y_n\hat{x})$$

$$= \hat{\mathbf{z}} \times \frac{\mathbf{p}_{n+1}+\mathbf{p}_n}{2} \tag{35}$$

and

$$\frac{\partial L_{n_z}}{\partial \mathbf{p}_n} = \frac{\mathbf{r}_{n+1}+\mathbf{r}_n}{2} = \hat{\mathbf{z}} \tag{36}$$

By utilizing (32), (33), (35), and (36) one easily obtains after a little algebra

$$[\varphi, L_{n_z}] = \frac{\partial \varphi}{\partial \mathbf{p}_n} \cdot \frac{\partial L_{n_z}}{\partial \mathbf{r}_n} - \frac{\partial \varphi}{\partial \mathbf{r}_n} \cdot \frac{\partial L_{n_z}}{\partial \mathbf{p}_n} = 0 \tag{37}$$

In the same manner one can demonstrate that the Poisson brackets of φ with the x and y components of $\mathbf{L}_n$ are equal to zero.

5. EXAMPLES

Let us consider the discrete equations of motion (2) and (3) of a particle of mass m starting at the time $t_0 = 0$ from $\mathbf{r}_0$ with momentum $\mathbf{p}_0$ and subject to the following potentials:

1. $V(\mathbf{r}) = 0$ (free particle)
2. $V(\mathbf{r}) = \alpha \mathbf{r} \cdot \hat{\mathbf{x}}$ (constant force)
3. $V(\mathbf{r}) = k\mathbf{r}^2/2$ (harmonic oscillator)
4. $V(\mathbf{r}) = c/r$ (gravitational force)

The discrete Hamiltonian is

$$H_n = T_n + V_n \tag{38}$$

where

$$T_n = \mathbf{p}_n^2/2m \tag{39}$$

We define [see equation (4)]

$$\dot{\mathbf{r}}_n = \frac{\mathbf{r}_{n+1} - \mathbf{r}_n}{t_{n+1} - t_n} \tag{40}$$

and

$$\dot{\mathbf{p}}_n = \frac{\mathbf{p}_{n+1} - \mathbf{p}_n}{t_{n-1} - t_n} \tag{41}$$

and assume equal time spacing $\Delta t_n = t_{n+1} - t_n = \varepsilon$. The equations of motion (2) and (3) thus became

$$\frac{\mathbf{r}_{n+1} - \mathbf{r}_n}{\varepsilon} = \frac{\mathbf{p}_{n+1} + \mathbf{p}_n}{2m} \tag{42}$$

$$\frac{\mathbf{p}_{n+1} - \mathbf{p}_n}{\varepsilon} = -\frac{\partial V_n}{\partial \mathbf{r}_n} \tag{43}$$

5.1. Free Particle

For a free particle of mass m the potential V_n is constant and one obtains from (43) and (42)

$$\mathbf{p}_n = \mathbf{p}_0 \tag{44}$$

and

$$\mathbf{r}_n = \mathbf{r}_0 + (\mathbf{p}_0/m)t_n \tag{45}$$

where $t_n = n\varepsilon$.

5.2. Constant Force

In this case

$$V(\mathbf{r}_n) = m\alpha \mathbf{r}_n \cdot \hat{\mathbf{x}} \tag{46}$$

and (42) and (43) become

$$\frac{\partial H_n}{\partial \mathbf{p}_n} = \frac{\mathbf{p}_{n+1} + \mathbf{p}_n}{2m} = \frac{\mathbf{r}_{n+1} - \mathbf{r}_n}{\varepsilon} \tag{47}$$

$$\frac{\partial V_n}{\partial \mathbf{r}_n} = m\alpha\hat{x} = -\frac{\mathbf{p}_{n+1} - \mathbf{p}_n}{\varepsilon} \tag{48}$$

Thus, for the y and z components we find the analogous one-dimensional equations of (44) and (45), while for the x component we obtain, with some algebra,

$$p_{n_x} = p_{0_x} - m\alpha t_n \tag{49}$$

$$x_n = x_0 + (p_{0_x}/m)t_n - \tfrac{1}{2}\alpha t_n^2 \tag{50}$$

Since

$$T_n = (p_0^2/2m) - \alpha p_{0_x} t_n + \tfrac{1}{2} m\alpha^2 t_n^2 \tag{51}$$

$$V_n = \alpha m x_0 + \alpha p_{0_x} t_n - \tfrac{1}{2} m\alpha^2 t_n^2 \tag{52}$$

the energy

$$E_n = T_n + V_n = p_0^2/2m + \alpha m x_0 = E_0 \tag{53}$$

is conserved.

5.3. Harmonic Oscillator

Substituting $V_n = k\mathbf{r}_n^2/2$ into equations (42) and (43), we obtain

$$\mathbf{r}_{n+1} - \mathbf{r}_n = (\varepsilon/2m)(\mathbf{p}_{n+1} + \mathbf{p}_n) \tag{54}$$

$$\mathbf{p}_{n+1} - \mathbf{p}_n = -\tfrac{1}{2}k\varepsilon(\mathbf{r}_{n+1} + \mathbf{r}_n) \tag{55}$$

We consider, for simplicity, a unique component of the coordinates and momenta, i.e., the x component. Equations (54) and (55) lead to

$$x_{n+1} - 2x_n + x_{n-1} + \tfrac{1}{4}\omega^2\varepsilon^2(x_{n+1} + 2x_n + x_{n-1}) = 0 \tag{56}$$

$$p_{n+1_x} - 2p_{n_x} + p_{n-1_x} + \tfrac{1}{4}\omega^2\varepsilon^2(p_{n+1_x} + 2p_{n_x} + p_{n-1_x}) = 0 \tag{57}$$

where, as usual,

$$\omega^2 = k/m \tag{58}$$

By including the initial conditions, we find for equations (56) and (57), respectively, the solutions

$$x_n = x_0 \cos n\varepsilon\nu + (p_{0_x}/\omega m)\sin n\varepsilon\nu \tag{59}$$

$$p_{n_x} = -x_0\omega m \sin n\varepsilon\nu + p_{0_x}\cos n\varepsilon\nu \tag{60}$$

where

$$\nu = \frac{1}{\varepsilon}\arcsin\frac{\omega\varepsilon}{1 + \frac{1}{4}\omega^2\varepsilon^2} \tag{61}$$

These results can be easily verified by substituting (59) and (60) into (56) and (57), respectively.

Thus, the discrete energy

$$E_n = \frac{\mathbf{p}_n^2}{2m} + V(\mathbf{r}_n) = \frac{\mathbf{p}_0^2}{2m} + \tfrac{1}{2}k\mathbf{r}_0^2 = E_0 \tag{62}$$

is conserved, while from (61) we see that the frequency of the discrete oscillator is different from the corresponding continuous frequency. For small time intervals, equation (61) reduces to

$$\nu = \omega\left(1 - \frac{\omega^2\varepsilon^2}{12}\right) + o(\varepsilon^4) \tag{63}$$

Thus, ν is less than ω.

These results have already been reported (D'Innocenzo *et al.*, 1986), where the consequences, in classical and relativistic mechanics, of a Lagrangian different from that used by Lee (1983) are discussed.

5.4. Gravitational Force

In this case $V_n = c/r_n$, and near equation (42), from (43) we have

$$\frac{\mathbf{p}_{n+1} - \mathbf{p}_n}{\varepsilon} = c \frac{\mathbf{r}_{n+1} + \mathbf{r}_n}{r_{n+1} r_n (r_{n+1} + r_n)} \tag{64}$$

This result was already proposed (Greenspan, 1974) as a definition of the discrete gravitational force.

The system formed by equations (42) and (64) can be solved numerically, for example, by using Newton's method.

6. CONCLUSIONS

Starting from a correspondence between continuous and discrete variables, we have proposed a simple model of discrete classical mechanics where the physical system evolution is governed by discrete Hamilton equations. Discrete gradients are defined in a proper manner, and thus the fundamental conservation laws remain valid. Different choices of the discrete gradient correspond to different discrete mechanical systems that have the same continuous limit. However, it is necessary that the various forms of discrete mechanics possess the symmetry properties of continuous mechanics. Our choice leads to very simple equations of motion, as we have shown with some examples.

REFERENCES

Clarke, F. H. (1975). *Transactions of the American Mathematical Society*, **205**, 247–262.

D'Innocenzo, A., Renna, L., and Rotelli, P. (1984). *Physics Letters B*, **145**, 268–270.

D'Innocenzo, A., Renna, L., and Rotelli, P. (1986). *European Journal of Physics*, to be published.

Friedberg, R., and Lee, T. D. (1983). *Nuclear Physics B*, **225**, 1-52.

Greenspan, D. (1974). *Bulletin of the American Mathematical Society*, **80**, 553–555.

Landau, L. D., and Lifshitz, E. M. (1969). *Mechanics*, Pergamon, New York.

Lee, T. D. (1983). *Physics Letters*, **122B**, 217–220.

Rockafellar, R. T. (1983). *Mathematical Programming. The State of the Art*, Springer-Verlag, New York, pp. 368–390.